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Self-Dual $SU(N)$ Gauge Fields

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We give a method by which we can construct solutions to the self-dual $SU(N)$ gauge field equations, some of which can be chosen as seed solutions.

1. INTRODUCTION

Belinskey and Zakharav (1978, 1979) established the theory of the soliton solutions for gravitational fields, and Letelier (1984, 1985), Zhong (1990), and Gao and Zhong (1992) developed it extensively. The field equations for the self-dual $SU(N)$ gauge fields have a similar form to the BZ equation for the gravitational field. Letelier (1986) studied the $SU(5)$ case.

In this paper we find that the seed condition for the gravitational BZ equation can be applied to the self-dual $SU(N)$ field equation. By studying the three seed solutions proposed by Letelier (1986), we see that they are block-diagonal, and we give a method by which a number of self-dual $SU(N)$ gauge field solutions can be constructed and chosen as seed solutions.

2. SEED SOLUTION CONDITION

For the axisymmetric vacuum gravitational field, the line element can be taken as

$$ds^2 = f^{-1}[e^{\tau}(d\rho^2 + dz^2) + \rho^2\, d\phi^2] + f(dt - \omega\, d\phi)^2 \qquad (2.1)$$

where f, ω, and τ are real functions of ρ and z, and τ is determined by f and ω. Consider the BZ equation

$$\partial_\rho[\rho\partial_\rho M(J)\cdot M^{-1}(J)] + \partial_z[\rho\partial_z M(J)\cdot M^{-1}(J)] = 0$$

$$\det M(J) = (-1)^J, \qquad M^T(J) = M(J) \qquad (2.2)$$

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where $J = 1, 2$, the symbol T denotes matrix transposition, and $M(J)$ is the 2×2 double matrix

$$M(J) = \frac{1}{F(J)} \begin{pmatrix} 1 & \Omega(J) \\ \Omega(J) & \Omega^2(J) + (-1)^J F^2(J) \end{pmatrix} \tag{2.3}$$

From the double solution of equation (2.3) we obtain a pair of gravitational dual solutions (f, ω) and $(\hat{f}, \hat{\omega})$:

$$\begin{cases} f = F(2) \\ \omega = V_{F(2)}(\Omega(2)) \end{cases} \qquad \begin{cases} \hat{f} = T(F(1)) \\ \hat{\omega} = \Omega(1) \end{cases}$$

$$T(F(J)) = \rho F^{-1}(J)$$

$$V_{F(J)}(\Omega(J)) = \int \rho F^{-2}(J)[\partial_z \Omega(J) \cdot d\rho - \partial_\rho \Omega(J) \cdot dz] \tag{2.4}$$

According to Gao and Zhong (1992), there are two seed solution conditions:

(i) The seed solution $M(J)$ satisfies

$$M_0(J) = M_0(\varphi_1, \ldots, \varphi_s; J) \tag{2.5a}$$

(ii) The seed solution $M(J)$ satisfies

$$\frac{\partial}{\partial \varphi_i} \left[\frac{\partial}{\partial \varphi_j} M_0(\varphi_1, \ldots, \varphi_s; J) \cdot M_0^{-1}(\varphi_1, \ldots, \varphi_s; J) \right] = 0 \tag{2.5b}$$

where $\varphi_1, \ldots, \varphi_s$ satisfies $\nabla^2 \varphi_s = 0$. The corresponding inverse scattering wave function can be directly obtained as

$$\psi_{0k} = M_0(\varphi_1 \to Y_{1k}, \ldots, \varphi_s \to Y_{sk}; J) \tag{2.6}$$

where the arrow denotes that φ_s is replaced by Y_{sk}, with

$$Y_{sk} = \frac{1}{2} \int [\rho/\mu_k(J)][\partial_\rho \mu_k(J) \cdot \partial_\rho \varphi_s - \partial_z \mu_k(J) \cdot \partial_z \varphi_s] \cdot d\rho$$

$$- \frac{1}{2} \int [\rho/\mu_k(J)][\partial_z \mu_k(J) \cdot \partial_\rho \varphi_s + \partial_\rho \mu_k(J) \cdot \partial_z \varphi_s] \cdot dz$$

$$\mu_k(J) = \alpha_k(J) - z \pm [\alpha_k^2(J) - 2\alpha_k(J)z + z^2 + \rho^2]^{1/2} \tag{2.7}$$

For the gravitational BZ equation, the seed solution $M(J)$ and the corresponding inverse function ψ_{0k} are 2×2 matrices. One can prove that the seed solution conditions (2.5a) and (2.5b) can be applied to the case of the self-dual $SU(N)$ gauge fields, in which $M(J)$ and ψ_{0k} are $N \times N$ matrices ($N \geq$

2) and J is merely the ordinary complex number unit, so that the $N \times N$ matrices M and ψ_{0k} are ordinary complex matrices.

3. SOLUTIONS TO THE SELF-DUAL $SU(N)$ GAUGE-FIELD EQUATIONS

For the self-dual $SU(N)$ gauge field equation

$$\partial_{\bar{\zeta}}(\partial_{\zeta} g \cdot g^{-1}) + \partial_{\bar{\eta}}(\partial_{\eta} g \cdot g^{-1}) = 0 \tag{3.1}$$

g is an $N \times N$ Hermitian matrix with unit determinant, $\det g = 1$, the variables ζ and η are the ordinary complex coordinates related to the four-dimensional Euclidean Cartesian coordinates, and $\bar{\zeta}$ and $\bar{\eta}$ are the complex conjugates of ζ and η.

Let $r = (2\zeta\bar{\zeta})^{1/2}$ and $z = (\eta + \bar{\eta})/\sqrt{2}$. Then equation (3.1) reduces to

$$\begin{aligned} &\partial_r(r\partial_r g \cdot g^{-1}) + \partial_z(r\partial_z g \cdot g^{-1}) = 0 \\ &g^{+} = g, \det g = 1 \end{aligned} \tag{3.2}$$

The soliton solutions of equation (3.2) are obtained by using

$$\begin{aligned} g_{ab} &= (g_0)_{ab} - \sum_{k,l} \bar{N}_a^{(l)}(\Gamma^{-1})_{lk} N_b^{(k)}/\mu_k\mu_l \\ \Gamma_{kl} &= m^{(k)}\bar{m}^{(l)}/(r^2 - \mu_k\bar{\mu}_l) = \bar{\Gamma}_{kl} \\ m^{(k)}\bar{m}^{(l)} &\equiv m_a^{(k)}(g_0)_{ab}\bar{m}_b^{(l)} \\ N_a^{(k)} &= m_b^{(k)}(g_0)_{ba}, \qquad m_a^{(k)} = m_{0b}^{(k)} M_{ba}^{(k)} \\ M^{(k)} &= \psi_0^{-1}\big|_{\lambda=\mu_k}, \qquad \mu_k = \alpha_k - z \pm [(\alpha_k - z)^2 + r^2]^{1/2} \\ \det g_n &= (-1)^n r^{2n}\left(\prod_{l=1}^{n} |\mu_l|^{-2}\right)\cdot \det g_0 \\ g^{ph} &= g_n/(\det g_n)^{1/N} \end{aligned} \tag{3.3}$$

where ψ_0 is a $N \times N$ complex matrix function of ρ and z, λ is a spectral parameter, and g_0 is a seed solution to equation (3.1) with the determinant $\det g_0 = \pm 1$.

For $N = 5$, Letelier gives the following three seed solutions:

$$g_0 = \mathrm{diag}(\eta_1 e^{\phi_1}, \eta_2 e^{\phi_2}, \eta_3 e^{\phi_3}, \eta_4 e^{\phi_4}, \eta_5 e^{\phi_5}) \tag{3.4a}$$

$$g_0 = \begin{pmatrix} \eta_1 e^{\phi_1} & 0 & 0 & 0 & 0 \\ 0 & \eta_2 e^{\phi_2} & 0 & 0 & 0 \\ 0 & 0 & \eta_3 e^{\phi_3} & 0 & 0 \\ 0 & 0 & 0 & \phi_4 & e^{iC_4} \\ 0 & 0 & 0 & e^{-iC_4} & 0 \end{pmatrix} \tag{3.4b}$$

$$g_0 = \begin{pmatrix} \eta_1 & 0 & 0 & 0 & 0 \\ 0 & \phi_2 & e^{iC_2} & 0 & 0 \\ 0 & e^{-iC_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_4 & e^{iC_4} \\ 0 & 0 & 0 & e^{-iC_4} & 0 \end{pmatrix} \tag{3.4c}$$

where the coefficients η_i $(i = 1, \ldots, 5)$ take the value ± 1. Here C_2 and C_4 are two arbitrary real constants; ϕ_i $(i = 1, \ldots, 5)$ satisfy $\nabla^2 \varphi_i = 0$, where the operator $\nabla^2 = \partial_r^2 + (1/r)\partial_r + \partial_z^2$ and $\det g = \pm 1$. By using the above three seed solutions (3.4), the corresponding inverse scattering wave function can be directly obtained. Considering the seed solution matrices (3.4b) and (3.4c) as supermatrices, we see that they are block-diagonal. Rewriting the formulas (3.4b) and (3.4c), we have

$$g_0 = \mathrm{diag}(R_1, R_2, R_3, R_4)$$

$$R_1 = \eta_1 e^{\phi_1}, \qquad R_2 = \eta_2 e^{\phi_2}, \qquad R_3 = \eta_3 e^{\phi_3}, \qquad R_4 = \begin{pmatrix} \phi_4 & e^{iC_4} \\ e^{-iC_4} & 0 \end{pmatrix} \tag{3.5a}$$

$$g_0 = \mathrm{diag}(R_1, R_2, R_3)$$

$$R_1 = \eta_1, \qquad R_2 = \begin{pmatrix} \phi_2 & e^{iC_2} \\ e^{-iC_2} & 0 \end{pmatrix}, \qquad R_3 = \begin{pmatrix} \phi_4 & e^{iC_4} \\ e^{-iC_4} & 0 \end{pmatrix} \tag{3.5b}$$

where the supermatrix elements R_i $(i = 1, 2, 3, 4)$ satisfy equation (3.2).

In conclusion, we write the seed solutions as

$$g = \sum \oplus R_i \tag{3.6}$$

where R_i are $i \times i$ matrices $(i = 1, \ldots, N - 1)$ which satisfy equation (3.1) and the determinant $\det R_i = \pm 1$.

In order to seed the solutions of equation (3.1), we construct the following supermatrix g:

$$\begin{pmatrix}
\eta_1 R_1 & & & & \bigcirc & & & & \eta_{2k}R_{2k} \\
& \ddots & & & & & & \iddots & \\
& & \eta_{k-1}R_{k-1} & & \circ & & \eta_{3k-2}R_{3k-2} & & \\
\bigcirc & & \circ & & \eta_k R_k & & \circ & & \bigcirc \\
& & \eta_{3k-2}R_{3k-2} & & \circ & & \eta_{k+1}R_{k+1} & & \\
& \iddots & & & & & & \ddots & \\
\eta_{2k}R_{2k} & & & & \bigcirc & & & & \eta_{2k-1}R_{2k-1}
\end{pmatrix} \tag{3.7a}$$

$$\begin{pmatrix}
\eta_1 R_1 & & & \bigcirc & & & \eta_{2k+1}R_{2k+1} \\
& \ddots & & & & \iddots & \\
\bigcirc & & \eta_k R_k & & \eta_{3k}R_{3k} & & \bigcirc \\
& & \eta_{3k}R_{3k} & & \eta_{k+1}R_{k+1} & & \\
& \iddots & & & & \ddots & \\
\eta_{2k+1}R_{2k+1} & & & \bigcirc & & & \eta_{2k}R_{2k}
\end{pmatrix} \tag{3.7b}$$

where the determinant of the supermatrix (3.7a) and (3.7b) is $\det g = \pm 1$, noticing that the matrix g for $\det g = -1$ is a nonphysical solution. η_i are real constants ($i = 1, \ldots, 3k$) and R_i are $i \times i$ matrices ($i = 1, \ldots, 3k$) which satisfy equation (3.1) and the determinant $\det R_i = \pm 1$.

For the case $N = 5$, we have three seed solutions. First,

$$g_0 = \begin{pmatrix}
\eta_1 e^{\phi_1} & 0 & 0 & 0 & \eta_6 e^{\phi_6} \\
0 & \eta_2 e^{\phi_2} & 0 & \eta_7 e^{\phi_7} & 0 \\
0 & 0 & \eta_3 e^{\phi_3} & 0 & 0 \\
0 & \eta_7 e^{\phi_7} & 0 & \eta_4 e^{\phi_4} & 0 \\
\eta_6 e^{\phi_6} & 0 & 0 & 0 & \eta_5 e^{\phi_5}
\end{pmatrix} \tag{3.8a}$$

where ϕ_i ($i = 1, \ldots, 7$) satisfy $\nabla^2 \phi_i = 0$ and $\phi_6 = (\phi_1 + \phi_5)/2$, $\phi_7 = (\phi_2 + \phi_4)/2$, $\Sigma_{i=1}^5 \phi_i = 0$, and η_i ($i = 1, \ldots, 7$) satisfy $\eta_3(\eta_1\eta_5 - \eta_6^2)(\eta_2\eta_4 - \eta_7^2) = \pm 1$. Second,

$$g_0 = \begin{pmatrix}
\eta_1 \phi_1 & \eta_1 e^{iC_1} & 0 & \eta_3 \phi_3 & \eta_3 e^{iC_3} \\
\eta_1 e^{-iC_1} & 0 & 0 & \eta_3 e^{-iC_3} & 0 \\
0 & 0 & \eta_4 & 0 & 0 \\
\eta_3 \phi_3 & \eta_3 e^{iC_3} & 0 & \eta_2 \phi_2 & \eta_2 e^{iC_2} \\
\eta_3 e^{-iC_3} & 0 & 0 & \eta_2 e^{-iC_2} & 0
\end{pmatrix} \tag{3.8b}$$

where ϕ_i $(i = 1, 2, 3)$ satisfy $\nabla^2\phi_i = 0$ and C_i $(i = 1, 2, 3)$ are arbitrary real constants which satisfy $C_3 = (C_2 + C_1)/2$, and η_i $(i = 1, 2, 3)$ are real constants which satisfy $\eta_4(\eta_1\eta_2 - \eta_3^2) = \pm 1$. Third,

$$g_0 = \begin{pmatrix} \phi_3 & 0 & 0 & 0 & e^{iC} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & h_+(\phi_1, \phi_2) & h_0(\phi_1, \phi_2) & 0 \\ 0 & 0 & h_0(\phi_1, \phi_2) & h_-(\phi_1, \phi_2) & 0 \\ e^{-iC} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.8c}$$

where ϕ_i $(i = 1, 2, 3)$ satisfy $\nabla^2\phi_i = 0$ and C is an arbitrary real constant; $h_0(\phi_1, \phi_2)$ and $h_\pm(\phi_1, \phi_2)$ are given as

$$\begin{aligned} h_0 &= [\phi_1/(\phi_1^2 + \phi_2^2)^{1/2}] \operatorname{sh}(\phi_1^2 + \phi_2^2)^{1/2} \\ h_\pm &= \operatorname{ch}(\phi_1^2 + \phi_2^2)^{1/2} \pm [\phi_2/(\phi_1^2 + \phi_2^2)^{1/2}] \operatorname{sh}(\phi_1^2 + \phi_2^2)^{1/2} \end{aligned} \tag{3.9}$$

According to formulas (3.8a) and (3.8b), we can construct the self-dual $SU(N)$ solutions for both N = odd and N = even. In fact, a number of solutions can be obtained and the solutions (3.8a)–(3.8c) can all be chosen as the seed solutions of equation (3.2), which satisfy the seed solution conditions (2.5a) and (2.5b). The corresponding inverse scattering wave functions similar to formula (2.6) can be obtained as

$$\begin{aligned} \psi_{0k} &= \psi|_{\lambda=\mu_k} = g(\phi_i \to Y_{ik}) \\ Y_{ik} &= \frac{1}{2}\int (r/\mu_k)(\partial_r \mu_k \cdot \partial_r\phi_i - \partial_z\mu_k \cdot \partial_z\phi_i)\, d\rho \\ &\quad - \frac{1}{2}\int (r/\mu_k)(\partial_r\mu_k \cdot \partial_z\phi_i + \partial_z\mu_k \cdot \partial_r\phi_i)\, dz \end{aligned} \tag{3.10}$$

where the arrow implies that Y_{ik} substitutes for ϕ_i.

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